Automated Study Of Envelopes:
The transition from 2D to 3D

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Abstract
Envelopes of parametric families of curves and of surfaces is a classical topic suffering from a non-well developed theory. Previous works studied 1-parameter families of plane curves, and showed how automated methods are efficient to revive the topic. In this paper, we analyse the transition from 2D to 3D, which is non-trivial, in particular because of visualization issues and of technological features available for 2D, but maybe not in a 3D setting. The study of envelopes enables also to make connections between different mathematical domains, such as between Differential Geometry and Abstract Algebra (Polynomial rings and their ideals). This provides an access to new algorithms implemented in CAS, and enhances automated methods of exploration and proof.

I.   Introduction
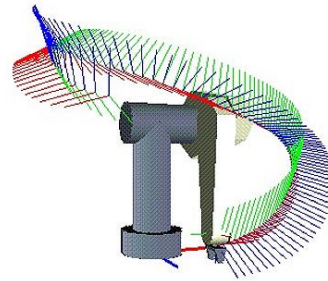### 1. The revival of envelopes

Envelopes of parametric families of curves or of surfaces are a classical topic in Differential Geometry, a mathematical domain called sometimes in the past Application of Calculus to Geometry. The study of such a topic helps to view mathematics not as a collection of disjoint domains, but as one domain with connected subdomains. Different mathematical topics are often studied in different courses, without any reference from one to the other; the study of envelopes shows bridges between two mathematical domains, Calculus and Geometry. It is important to build such bridges: intuition on methods to work out problems may come from a broad knowledge within specific domains and about bridges between them. As Artigue [3] explains, even if the general aim of mathematics education is not to educate only future researchers, working while integrating different domains is efficient to broaden horizons and enhance mathematical culture.

The theory of envelopes has numerous applications in real life such as caustics and wave fronts (geometrical optics, theory of singularities), robotics (rigid motion in 2-space and in 3-space, collision avoidance, whence their importance in the design of industrial plants), diffusion of chemicals in the soil, soil depollution, etc. For example, Figure 1a shows a sprinkler for lawn irrigation. The envelope of the trajectories of water drops is the borderline separating the zone where one can remain dry from the zone where one will be wet. Figure 1b explains how the surface envelope of the family of surfaces generated by the end of arms of the robot defines the safety zone, where the arms will not hurt anybody. Back in 1962, Thom [27] had complained on the disappearance of envelopes from the curriculum and emphasized the role of envelopes in the theory of Differential Equations (singular equations) and in Partial Differential Equations. He wonders if it is "conceivable" for a teacher to teach problems of Elementary Geometry without any knowledge of

the general phenomena of the theory. Thom notes that even from a practical point of view, the theory of envelopes reports on familiar phenomena, which could have no explanation without it. "It is enough to look at a hemispheric cup of coffee with suitable lighting to see the cuspidal structure of a caustic and its variations when lighting changes".

<table>
<tr><td>(a)</td><td>(b)</td></tr>
</table>

Figure 1: *Surface envelopes in real life*

2. The role of software in the revival

According to Thom, among the possible reasons for the disappearance of the theory of envelopes from the curriculum, is that it is not rigorous enough, and that it has few powerful theorems and numerous special cases. Therefore, central issues are as follows:

- Computation abilities (described in the next section) are important to solve the systems of equations describing the envelopes. For algebraic curves, i.e. with polynomial data, software packages based on the computation of Gröbner bases and on elimination are a suitable tool. Non-polynomial data may sometimes be transformed into polynomial form. In both cases, either polynomial or non-polynomial equations, the software command may be the same, namely **solve**, but it uses pattern recognition to decide which algorithm to apply (see [16]).
- Visualization of the envelope (if it exists) is important when doing experimental work. Thom mentions the cuspidal structure of many envelopes, which is not always easy to see on the screen, especially in 3D. Details are found in Section III.

3. Registers of representation

In [19], Duval pointed out that there is a basic difference between mathematics and the other domains of scientific knowledge. In contrast to phenomena of astronomy, physics, chemistry, biology, etc., mathematics used not to be accessible by perception or by instruments (microscopes, telescopes, and measurement apparatus). Mathematical objects are abstract entities, accessible only via various representatives. Duval [18] identified three registers *of representation*, numerical, graphical and algebraic, pointing out to the importance of the ability to switch from one register to another. Technology offers different representatives of the same object, hence facilitating exploration.

Plotting a graph after entering an algebraic expression (a function, an implicit equation), is a switch from the algebraic register to the graphical one[1]. Whereas switching in the reversed direction is generally not available in a Computer Algebra System (CAS), Dynamical Geometry Systems (DGS) such as GeoGebra make a graphics-to-algebra switch available. For example, GeoGebra

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[1] Actually, the computer performs a first translation from an algebraic representation to a numerical one, then plots the curve/the surface, performing a second translation from numerical to graphical representation. In this paper, switching between algebra and graphics is of interest.

translates immediately the geometrical constructions performed with the mouse in the graphical window into coordinates and equations in the algebraic window. This is an important feature for conjecturing and then obtaining symbolic results. An important application is the determination of geometric loci, using the **Locus** and **LocusEquation** commands, (the algorithms are described in [22]).

The intertwining of experimental approaches and theoretical thinking is the core of discovery in every mathematical field ([24], [7]). Leikin and Grossman note in [23] that "Mathematics education researchers studying inquiry-based learning use terms such as 'investigation', 'exploration', and 'experimentation', while the boundaries between these terms are rather vague… Therefore … we consider investigations in geometry as activities that include experimenting (to arrive at a conjecture), conjecturing, testing (the conjecture), and proving (or refuting) it." In our present work, the dynamical features of the DGS, especially the continuous dragging of a point using the mouse and/or the availability of a slider bar, help to experiment and express conjectures, while an algebraic package of the CAS is a suitable tool to prove the conjecture.

4. The transition from 2D to 3D

The different software packages have different graphical affordances in 2D and in 3D. In particular, some dynamical features, which are available in 2D, are still inexistent or only under development in 3D. As we will see later, the existence of a slider bar enabling controlled changes in parameter values is very important for dynamical exploration, but is not always available in 3D. We should mention also that the continuous dragging of points on the screen and its effects are easier to understand in 2D than in 3D. The algebraic components of a CAS are however well developed, and actually, the same algorithms seem to be at work in 2D and 3D. Nevertheless, in both settings, the higher the dimension the heavier the computations, which can become unilluminating.

In this paper, we show how the envelopes can be studied making a joint usage of both a CAS and a DGS. The DGS provides an environment for experiments, from which conjectures can be obtained. The CAS enables to actually prove or disprove the conjectures, using analytic and algebraic methods. Nevertheless, the transition from 1-parameter families of plane curves to 1-parameter families of surfaces in 3D space presents a double trend. Whereas the algebraic part follows a similar pathway in both settings, the visualization is different.

Section II of this paper is devoted to a survey of previous results on 1-parameter families of curves and emphasizes the efficiency of work using together CAS and DGS. Section III presents a study of a 1-parameter family of surfaces and Section IV is devoted to a general discussion of the non-trivial transition from 2D to 3D.

This paper is an extended version of a communication at INDRUM 1 conference, 2016 [14].

II. Envelopes of 1-parameter families of plane curves

Let be given a family of plane curves be given by the equation $f(x, y, t) = 0$, where t is a real parameter. An envelope of the family, if it exists, is a curve tangent to every curve in the family (see for example [26]). This envelope is the solution set of the following system of equations (op. cit., p. 167):

$$(1) \qquad \begin{cases} f(x, y, t) = 0 \\ \dfrac{\partial f}{\partial t}(x, y, t) = 0 \end{cases}.$$

Figure 2 below (plotted using GeoGebra[2]) shows the envelopes of two families of circles centred on the same parabola, whose equation is $y = \frac{1}{4}x^2$. In Figure 2a, the circles have a radius equal to 1, and in Figure 2b, the radius is equal to 3.5. The visualization suggests immediately that in both cases an envelope exists, and that it is the union of two disjoint components. Actually, each of them is called an envelope ([26], [6]). Moreover, there may be a difference between the two cases: in Figure 2a, it seems that both components are smooth, but in Figure 2b, singular points or cusps seem to exist. Using CAS and theorems from Advanced Calculus, equations for the envelopes can be found and the existence of singular points and their specific nature can be established.

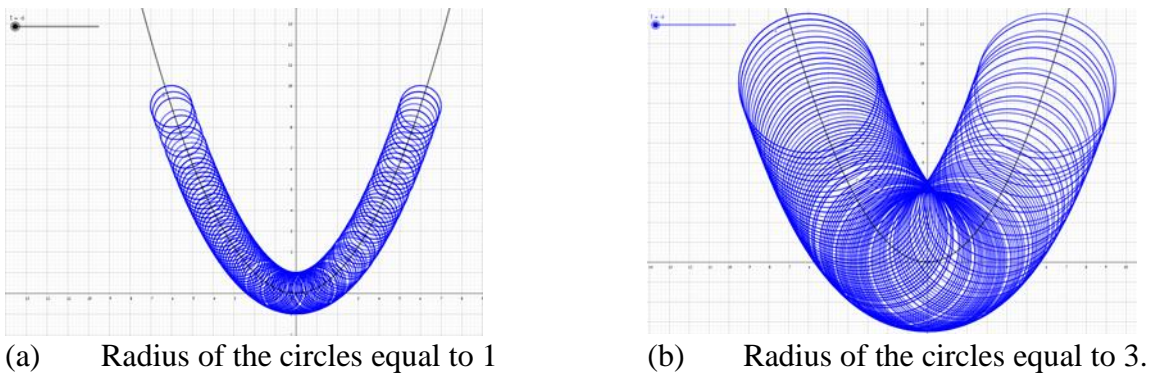(a)   Radius of the circles equal to 1   (b)   Radius of the circles equal to 3.

Figure 2: *Envelopes of two families of circles centred on the same parabola*

In these figures, the parabola has been defined with an implicit equation, and the family of circles by the presentation $(x - t)^2 + \left(y - \frac{t^2}{4} \right)^2 = R^2$, where R is the specific radius. The DGS understands that t is a parameter and proposes to create a slider bar (in the upper-left corner of the display). In order to have more particular cases for the circles, we could have created a slider bar for R too. This is useful for an experimental checking that the existence of singular points depends on the radius of the circles. Using the **Trace On** option for the lines and moving the cursor along the slider bar, a possible envelope appears.

The same kind of figure can be obtained without the usage of a slider bar. The parabola is plotted and a point A is marked on it. A circle is defined, whose center is A and radius is equal to 1. Then the point A is moved along the parabola with the **Trace On** option. The quality of the plot may be different. As an example, we invite the reader to compare the two plots for a nephroid in [12] (Figures 14 and 15b), as an envelope of a family of circles.

These examples show how exploring envelopes dynamically can enhance the understanding of various properties, among them:

a.   An envelope may not exist.
b.   An envelope may exist, and have more than one component.
c.   If an envelope exists, singular points may exist.

As noted above, a rigorous proof is needed for the existence of an envelope and for checking the existence of singular points. We show it now for the first case (Figure 2a), using a Maple code (see [12] and [15]).

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[2] See the GeoGebra applet https://www.geogebra.org/m/JxZ8cmBx .

A general point on the given parabola has coordinates $\left(t, \dfrac{t^2}{4}\right)$, whence a general equation

for the circles of the family is $(x-t)^2 + \left(y - \dfrac{t^2}{4}\right)^2 = 1$. Let $f(x,y,t) = (x-t)^2 + \left(y - \dfrac{t^2}{4}\right)^2 - 1$. In this

case, System (1) reads as follows:

$$\begin{cases} x^2 - 2xt^2 + y^2 - \dfrac{t^2}{2}y + \dfrac{t^4}{16} + t^2 - 1 = 0 \\ 2x + yt - \dfrac{t^3}{4} - 2 = 0 \end{cases}.$$

Using the **solve** command of the CAS, we obtain:

$$(x,y) = \left(t - \dfrac{1}{\sqrt{4t^2+1}}, \dfrac{t^2}{4} + \dfrac{8}{\sqrt{4t^2+1}}\right) \text{ or } (x,y) = \left(t + \dfrac{1}{\sqrt{4t^2+1}}, \dfrac{t^2}{4} - \dfrac{8}{\sqrt{4t^2+1}}\right).$$

These parametric presentations define two disjoint components; we show them in Figure 3), together with a few circles from the family.
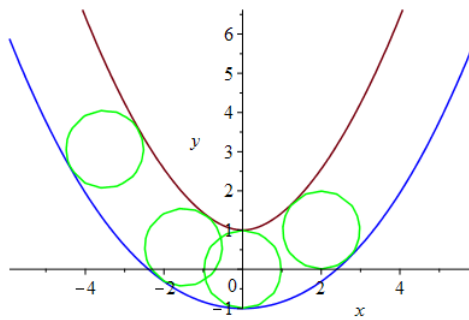

Figure 3: *Envelope of a family of unit circles centred on a parabola*

Elimination of the parameter in order to obtain an implicit equation requires the usage of a specific algebraic package of the CAS. Maple code for such an algebraic session is given in [12]; see also [13] and [15]. In this last paper, another transition is studied, namely from 1-parameter families to 2-parameter families of surfaces.

III.   Envelopes of 1-parameter families of surfaces.

The transition to parametrized families of curves and surfaces in 3D space rely on the same techniques and on a generalization of Equations (1). New issues have to be addressed, namely general visualization problems, and availability of appropriate features in the software, etc. Consider a family of surfaces given by an equation of the form $F(x, y, z, t) = 0$, where t is a real parameter. If an envelope exists, it is determined by the solutions of the system of equations

(2)
$$\begin{cases} F(x, y, z, t) = 0 \\ \dfrac{\partial F}{\partial t}(x, y, z, t) = 0 \end{cases}.$$

Example 1: A family of spheres

Consider the family of spheres of radius 5 centred on the line L whose equation is $y = x$ in the xy-plane. A general equation for this family of spheres is given by the polynomial

$$F(x, y, z) = (x - t)^2 + (y - t)^2 + z^2 - 25$$

A first experimentation with GeoGebra provides the plot in Figure 4. Intuitively, the envelope is a cylinder. To ascertain the intuition, we use Maple for the algebraic treatment of the problem.
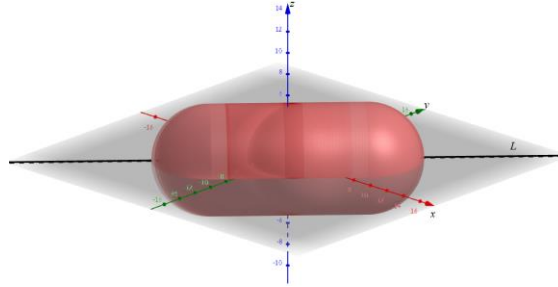

Figure 4: *A first approach to see the envelope of a family of spheres*

For the present case, System (2) reads as follows:

$$\begin{cases} (x - t)^2 + (y - t)^2 + z^2 - 25 = 0 \\ x + y - 2t = 0 \end{cases}$$

Its solution is given as two parametrizations, namely:

$$(3a) \qquad (x, y, z) \ = \ \left(2t - s,\ s,\ \sqrt{-2t^2 + 4ts - 2y^2 + 25}\right)$$

and

$$(3b) \qquad (x, y, z) \ = \ \left(2t - s,\ s,\ -\sqrt{-2t^2 + 4ts - 2y^2 + 25}\right)$$

We use an automated method described in [15] to find the envelope, which has been determined to be given by Equation (4):

- Transform the parametrization in Equation (3a) into polynomials. This requires squaring for the third component; therefore, the same result would have been obtained from Equation (3b).
- These polynomials generate an ideal K.
- Using the command **EliminationIdeal**, we obtain an ideal generated by the polynomial

$F(x, y, z) = x^2 - 2xy + y^2 + 2z^2 - 50$, i.e. the envelope is determined by the following equation:

$$(4) \qquad\qquad x^2 - 2xy + y^2 + 2z^2 - 50 = 0$$

This is the equation of the cylinder with axis L and basis any circle of radius 5 in a plane perpendicular to L. Note that the xy-part of the polynomial describes the intersection of the cylinder with the plane whose equation is $z = 0$. The plane is not perpendicular to the axis of the cylinder, and the intersection is an ellipse.

Intuition and visualization have been enhanced in Figure 5 (obtained with Maple) by using three steps: (a) some spheres of the family, (b) a partial plot of the cylinder enclosing some of the

spheres. In more complicated cases, this way of displaying may help, as transparency may be not enough to fully understand the picture.
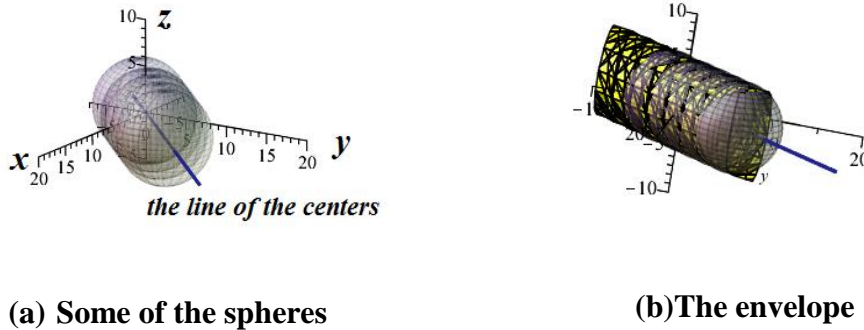

(a) **Some of the spheres**

(b)**The envelope**

Figure 5: *Envelope of a family of spheres*

Example 2: A second family of spheres

Consider now the family of spheres of radius 5 centred on the parabola whose equation is $4y = x^2$ in the *xy*-plane. Figure 6 shows the display on the screen obtained with a 3D animation in GeoGebra. Figure 6a shows a general view, and Figure 6b shows a view from above. This example can be understood as a 3D generalization of the example found in Section II, therefore it was expected that singular points would appear, the same way as cusps appeared in the 2D example.
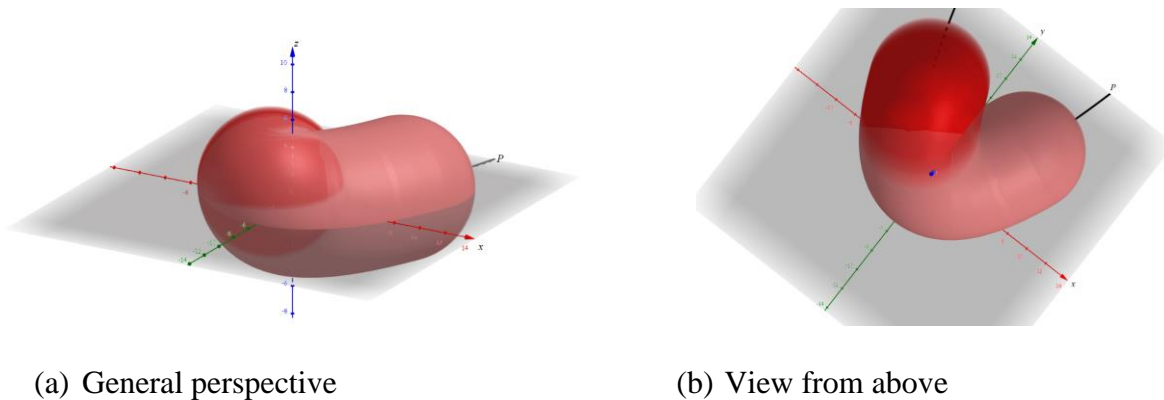

(a) General perspective   (b) View from above

Figure 6: A *family of spheres centred on a parabola*

It is harder to "see" these possible singularities. Therefore, an immediate switch to algebraic treatment with a CAS is needed.

System of Equations (2) reads here as follows:

(6)

$$\begin{cases} (x-t)^2 + \left(y - \dfrac{t^2}{4}\right)^2 + z^2 - 25 = 0 \\ 2x + ty - \dfrac{1}{4}t(t^2 + 8) = 0 \end{cases}$$

As in the previous example, Maple's **solve** command (followed by the **allvalues** command) yields two parametrizations, namely:

$$\left(x, y, z\right) = \left(\frac{1}{8}t^3 - \frac{1}{2}ts + t, s, \frac{1}{8}\sqrt{-t^6 + 8t^4 s - 4t^4 - 16t^2 s^2 + 32t^2 s - 64s^2 + 1600}\right)$$

and

$$\left(x, y, z\right) = \left(\frac{1}{8}t^3 - \frac{1}{2}ts + t, s, -\frac{1}{8}\sqrt{-t^6 + 8t^4 s - 4t^4 - 16t^2 s^2 + 32t^2 s - 64s^2 + 1600}\right)$$

which represent respectively the part of the requested envelope above the *xy*-pane and beneath the *xy*-plane. The obtained plot is shown in Figure 7.
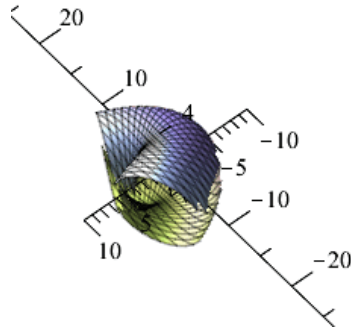

Figure 7: *a Maple parametric plot of the envelope*

We chose a view almost from above in order for the possible singularities to appear. Note that the plot should have been continuous on the "sides", but is not because of technical issues; we refer to [33] and [34] for their study.
We next derive the following Cartesian equation for the envelope:

$$x^6 + x^4 y^2 + 3x^4 z^2 + 2x^2 y^2 z^2 + 3x^2 z^4 + y^2 z^4 + z^6 - 10x^4 y - 8x^2 y^3 - 2x^2 yz^2$$
$$+ 8y^3 z^2 + 8yz^4 - 74x^4 - 18x^2 y^2 - 130x^2 z^2 + 16y^4 - 42y^2 z^2 - 83z^4 + 42x^2 y$$
$$- 232y^3 - 432yz^2 + 1375x^2 + 441y^2 + 2291z^2 + 5800y - 21025 \qquad = 0$$

Using the **implicitplot3d** command with the "style=patchnogrid" option, we obtain the surface in Figure 8. Note that here the surface is complete, without the "side-holes" of Figure 7.
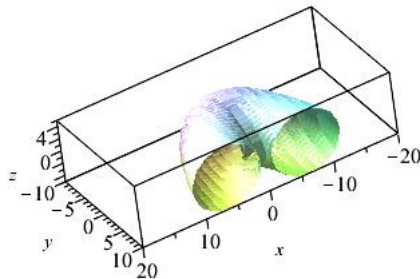

Figure 8: *an implicit plot of the envelope*

Example 3: A family of planes

A family of planes is given by a general equation $u(t)x + v(t)y + w(t)z + 1 = 0$, where t is a real parameter. In this case, System (1) reads as follows:

(7)
$$\begin{cases} u(t)x + v(t)y + w(t)z - 1 = 0 \\ u'(t)x + v'(t)y + w'(t)z = 0 \end{cases}.$$

This system has either no solution (for example when u, v, w are affine functions of the parameter t) or the solution determines a line in space. This last case means that an envelope exists and is generated by lines, i.e. the envelope is a *ruled surface.*

We recall that a ruled surface is a surface generated by straight lines; the best-known ruled surfaces are one sheet hyperboloids (Figure 9a) and hyperbolic paraboloids (Figure 9b). The figures below have been obtained with Maple. Note that the mesh chosen by the software enhances the generating lines. This is a valuable property of parametric plotting; see [34].
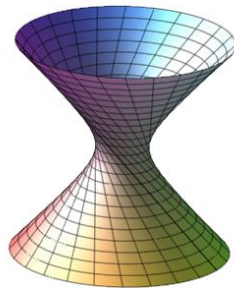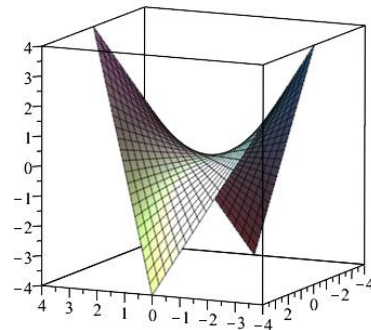
a. One sheet hyperboloid   b. Hyperbolic paraboloid

Figure 9: *Two ruled surfaces*

With respect to this, Ferréol mentions in [20] important properties of an envelope of a 1-parameter family of planes:

    a. It presents a *cuspidal edge* (called also an *edge of regression*), i.e. a curve whose points are singular points;

    b. The tangents to the cuspidal edge are generators of the envelope;

    c. The cuspidal edge is an envelope of the family of generators

Consider the family of planes given by the equation $tx + y + t^2 z = 2t^3$ where t is a real parameter. This family has an envelope, determined by the following system of equations

$$\begin{cases} tx + y + t^2 z - 2t^3 = 0 \\ x + 2tz - 6t^2 = 0 \end{cases}.$$

This system has the following solution:

(11)
$$(x, y, z) = \left(-2st - 6t^2, st^2 + 4t^3, s\right).$$

Eliminating the parameter yields an implicit equation:

(12)
$$x^2 z^2 - 4yz^3 - 8x^3 + 36xyz - 108y^2 = 0.$$

The envelope of the given family of planes is displayed in Figure 10 (obtained with Maple). The cuspidal edge appears clearly. Note that the mesh chosen by the software involves the tangents to the edge of regression.

This space curve may be studied using methods from Advanced Calculus: if a surface is given by a multivariate polynomial, the vanishing points of the first partial derivatives are singular points of the surface (see [2], page 696). Solving the system of equations $\dfrac{\partial F}{\partial x} = \dfrac{\partial F}{\partial y} = \dfrac{\partial F}{\partial z} = 0$ yields a parametric presentation of the cuspidal edge. Of course, this is doable with a CAS.

In order to understand the surface visually, the dynamical features of the software are important such as to rotate the plot using the mouse. Figure 10 shows two stills pictures in order to enable a better understanding. In particular, the edge of regression and its tangents can be identified. Note that the usage of software reveals the structure of the envelope as a ruled surface. In particular, the display shows that the tangents to the cuspidal edge are generators of the ruled surface.
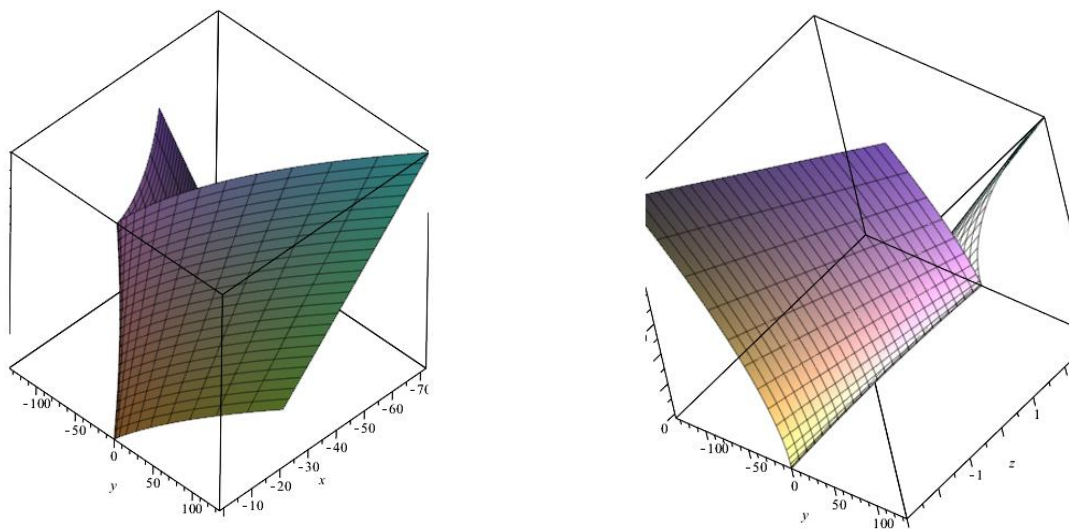

Figure 10: *Envelope of a family of planes: two different*

IV.   Discussion
1. CAS literacy

The first issue is the *choice of commands*. Plotting plane curves can be performed either with an implicit equation or with a parametric presentation. For the first case, Maple provides an **implicitplot** command, with a syntax like implicitplot(*implicit equation, x=x0..x1,y=y0..y1, options)*. For the second one, the **plot** command has to be used, with a specific format, namely *plot ([x(t),y(t),t=min..max])*, different from the format for plotting graph of a function of one real variable.

Space curves are given either by a parametric representation or as the intersection of two surfaces, i.e. by a pair of implicit equations. In such a case, the two surfaces can be plotted together using the command **implicitpot3d**, but identifying their intersection is almost impossible. A more accurate plot is obtained using a parametric presentation. For this Maple has two commands **plot3d** and **spacecurve**, which have a similar syntax but provide slightly different outputs (e.g. the default colour, if no specific option is given)

A second issue belonging to CAS literacy is how to perform *implicitization and elimination of the parameter*. Automated reasoning is based on dynamical geometrical features and algebraic algorithms. Visualization has a real appeal, but the process we follow to work with a CAS to solve

equations is not less important. Both in 2D and in 3D, the solution of the main system of equations is obtained in parametric form. It happens that an implicit equation is requested. It is obtained by elimination of the parameter, an operation often made possible within a specific package. Nevertheless, recall that parametric plot yields a display of higher quality than implicit plot, as the first one chooses a mesh according to specific properties of the surface, when the second one uses standard mesh, as explained in [33] and [34].

The algebraic engine to solve polynomial equations and systems of polynomial equations in different CAS is generally based on computations of Gröbner bases, a domain belonging to abstract algebra (ring theory). We refer to the books by Adams and Loustaunau [1] and Cox, Little and O'Shea [10].

Adams and Loustaunau provide a nice scaffolding to learn the topic, first leading hand-made computations, and then enabling to understand the algorithms implemented in software. Automated proofs using the software come only after that. The implementation enables to solve systems of equations, yielding a parametric representation of the solutions. Some packages provide also a tool for implicitization of this parametric representation.

The Gröbner bases algorithms have been widely used in Pech's book [25] for a new point of view on classical problems in geometry[3]. The usage of these algorithms for the study of envelopes and implicitization is described in [12] and [13].

In order to solve the system of equations (either (1) or (2)) defining an envelope we used the **solve** command of the CAS. The first performed step, non-visible to the user, is what is called *pattern recognition*. For our concern here, pattern recognition is a branch of machine learning that focuses on the recognition of patterns in data. For the **solve** command, the CAS analyzes the input and recognizes its nature, whether the equations are polynomial or not. If the input is polynomial, the CAS will work a described above. Otherwise, the CAS may give a solution, but via other algorithms, and the user may not be aware of the actual situation.

2. Interaction between mathematical domains

The title of classical textbooks for the study of curves and surfaces read as "Applications of Analysis to Geometry" (e.g. [9], Vol. 4, which was widely used in France in the past). The examples in this paper show actually an interaction between three domains, namely Geometry, Analysis and Algebra, the last one being mostly polynomial rings and their ideals (theory of Gröbner bases and applications).

The example of a 1-parameter family of planes showed an interesting property beyond the double trend Geometry-Analysis. The envelope of a 1-parameter family of planes is a ruled surface. Ruled surfaces used to be an important achievement of the curriculum in Linear Algebra (see Dieudonné's textbook [17]), but has generally been put aside for many years, despite its applications in civil engineering. In the same spirit as in Thom's remark for envelopes in [27], we have here an invitation to revive a classical topic.

Because of this interdisciplinary aspect, the study of envelopes of parametrized families of curves and surfaces provides an opportunity to discover new topics beyond the scope of the regular curriculum, sometimes together with applications to practical situations:

- Connections between mathematical domains may be discovered and explored.

- New computational skills with technology may be developed, in particular for the experimental aspect of the work (conjecturing the existence of an envelope and its properties,

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[3] Pech uses a software specifically designed for Gröbner bases computations, the CoCoA System, freely downloadable from cocoa.dima.unige.it

exploring the existence of cusps, as in Figure 1b). For this, the availability of a slider bar, together with a **Trace on** option, is a central issue.

A great advantage of a CAS is its multi-purpose structure. As a standard, a CAS enables switches from the algebraic register of representation to the graphical one, but not in the reversed direction. GeoGebra has the reversed switching, even when the slider bar is in use: the graphical display is automatically translated into coordinates and equations, in a separate window.

3. The transition from 2D to 3D: common difficulties and new constraints

Students learn curves given by a parametric presentation in an Advanced Calculus course. The transition towards surfaces is more seldom. A frequent pitfall is students' difficulties to understand these curves and surfaces. Kaur and Wangler [21] propose a way of working in order to "demystify parametric equations" (note that they speak about surfaces, but their remark is valid for curves also)**:** "In order to better understand the significance and role of each parameter, we consider how modifying parameter values impacts on the shape of the surface, we plot some of the grid curves to see how they mesh …, and we guide the students through several other explorations, where they first modify the parametric equations, then plot the graph, and finally, interpret what just happened".

This is the way many middle school and high school teachers explain, for example, the influence on the shape of a parabola of the parameters in its equation. Nowadays, this experimentation may be performed using DGS for plane curves. We emphasized previously the importance of the slider for this dynamical experimentation[4]. Our study of envelopes of families of surfaces provides an implementation of the same methods at a higher level.

Visualization in 3D is an important issue, and the non-availability of a slider in 3D makes dynamical experimentations harder to perform than in 2D. The affordances and constraints of the software are an important part of the attitude of the teacher, whether and how to rely on technology to convey such a new mathematical knowledge. Even when a slider bar is available, visualization is less clear in 3D than in 2D, whence conjectures may be harder to elaborate. The first two examples of Section III (families of spheres) enhanced the limitations of visualization for conjecturing the existence of an envelope, and then for conjecturing the shape of the envelope and the existence of singularities.

Trouche ([30], page 145) describes three types of constraints: *internal constraints* intrinsically linked to the material, *command constraints* linked to the various commands, their availability and syntax, and *organization constraints* linked to keyboard and screen organization. In [11], *motivating constraints* are added, linked to a lack of mathematical knowledge among the students: constraints may incite the user to learn more mathematics. In [28], Thomas and Hong emphasize this fact, together with the need for educator's *mathematical knowledge for teaching*. Bass et al. study also this last issue, in [5]; we meet here this question at undergraduate level. Because of visualization constraints, the efficiency of a slider for 2D seems much better than in 3D.

The exploration of envelopes requires the plotting of a large number of elements of the family in order to be able to conjecture either the existence or the non-existence of an envelope. Plotting on a screen is the "true" situation when dealing with plane curves, but for surfaces in 3-diemnsional space, the screen provides a projection on a plane only, and the user has to imagine what the spatial reality is. The overlapping of different elements of the family causes many difficulties for identifying the needed properties. This makes the transition from 2D to 3D a critical transition, according to Yerushalmy's terminology ([31], page 37).

4. Man-and-machine communication

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[4] See the GeoGebra applet at https://www.geogebra.org/m/dCZenyKw for the study of the shape of a parabola.

Visualization of geometrical objects is not the only issue here. The study of envelopes requires working at the same time in two registers of representation. In the previous paragraph, we evoked the transition from the geometrical point of view. Regarding the algebraic register, the transition from 2D to 3D is less critical, as System (2) is of a nature similar to System (1). The algorithms used by the CAS in order to solve them are the same algorithms. If the given equations are polynomials, then these algorithms are based on computations of Gröbner bases and elimination ideals. Otherwise, the same high-level commands are used (solve, simplify, etc.), but they request other options and run other algorithms.

According to the specific CAS and to the option used for the **solve** command, more than one command may have to be used in order to have a display using elementary functions. If *F* is a polynomial of degree higher than 2, the automated solution of the equations involves often the computer function *RootOf*. In order to overcome this, an additional option has to be entered for the solution display. In Maple, the **allvalues** command transforms this *RootOf* into expressions, which can be applied further. As this is generally not a part of the standard technological knowledge of the students, the teacher as to provide this knowledge. This belongs to the teacher's *technological discourse* (according to Artigue in [3]).

This makes the *mathematical knowledge for teaching* more important for the algebraic aspect than for the geometrical aspect.

We saw that experiments in the study of envelopes require a lot of communication:

- Man-and-machine, whence the importance of mastering the different registers of representation,
- Machine-and-machine, for the joint usage of a DGS (experimentation, explorations) together with a CAS (algebraic treatment of the data and rigorous proof of the results).

Trouche noted in [29] that the communicative dimension of the automated tools was becoming more and more important, about both man-and-machine communication (screens, input\output protocols, etc.) and machine-to-machine communication. In fact, Botana and Valcarce have postulated in [8] the increasing abilities of CAS and DGS to communicate. In [4], Arzarello et al. mentioned that the latest developments in these technologies have "rejuvenated mathematics and mathematics education", as they provide numerous new computational and symbolic tools, and revitalize experimental mathematics and visual tools. This rejuvenation appears in this paper through the possible revival of the study of envelopes in a software-rich environment. Increasing abilities of software and rejuvenation of mathematical domains come together.

The man-and-machine communication follows paths slightly different when passing from 2D to 3D. The dynamics provided by the slider bar enables a kind of dialog between the user and the technology. This dialog is harder, less dynamical when working in 3D. Actually, a 3D graphical representation provided by a DGS is a representation on a flat screen, a projection on a plane of a 3-dimensional object. Zooming and revolving the plot is available with every CAS, but not always reveals what has to be seen. We could expect to improve the 3D exploration via zooming in order to have a closer look at specific features, for example to enhance a better understanding of the cuspidal edge of a surface. This is not actual, as zooming inflates the cells and generally does not reveal hidden features. A solution to this problem could come from virtual reality, with the development of algorithms re-computing everything at every step ([33], [34]). Of course, this is CPU consuming.

Finally, one of the main problems is the accuracy of the plots. The holes in Figure 7 are an example of this issue, which has been studied in [32]. We saw in Section II different 2D plots using different methods offered by the DGS. In 3D, the accuracy problems are more important, as it appeared in Figures 9 and 10. The usage of different plot commands may help.

8. References

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